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ORIGINAL RESEARCH

Coupled coincidence point results in partially ordered generalized fuzzy metric spaces with applications to integral equations

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Abstract Here we prove a coupled coincidence point theorem in G-fuzzy metric spaces for compatible mappings using *Hadžić* type t-norms which is characterized by the equi-continuity of its iterates. We apply our result toward obtaining a result in G-metric spaces. Two supporting examples are also given. Some existing results are extended by our theorem. We also apply our result to a problem of an integral equation. We further assume the G-metric space to be equipped with a partial ordering.

Keywords Partially ordered set · G-fuzzy metric space · t-Norm of *Hadžić* · Compatible mappings · Coupled coincidence point · Integral equation

Introduction

The program of this work is to establish coupled coincidence point results in generalized fuzzy metric spaces which are actually fuzzy extensions of generalized metric spaces (abbreviated as G-metric spaces in the literatures). These spaces were introduced in the paper by Mustafa et al. [16, 17]. The generalization is effectuated by a non-negative real-valued mapping on X^3, where X is a given non-empty set on which the generalized metric is defined. Fixed point results on this structure were proved in a good number of papers, as, for instance [1, 5, 8, 10, 19].

Fuzzy sets were introduced by Zadeh [23] which provided an approach to non-probabilistic uncertain situations. Several fuzzified versions of the exiting mathematical structures were introduced in the literatures, particularly the fuzzification of metric space followed through adoption of different approaches. The flexible structure of fuzzy ideas allow for adopting different approaches resulting into the definitions of fuzzy metric spaces which are not equivalent to each other. Here we work on the definition given by George et al. [9] in which the topology is a Hausdorff topology. The fuzzy fixed point theory has a commendable development in the context of this space. One of the possible causes for that is the nature of the topology which is Hausdorff. In the theory of fuzzy fixed points and related topics on the above mentioned space. Some important references, amongst others, are [6, 7, 11].

Putting together the concepts involved in the two above mentioned spaces, generalized fuzzy metric spaces were introduced by [22]. Works on fixed point theory on this space are obtainable in [13, 18, 22].

This paper aimed at establishing a new coupled fixed point theorem in G-fuzzy metric spaces with a partial order. For this purpose we prove a lemma which establishes a Cauchy criterion for two sequences simultaneously. Hadzic type t-norm is used in this paper which is characterized by the equi-continuity of iterates. It is used in the proof of a lemma. We apply the result in this space to obtain new coupled fixed point results in G-metric spaces. Finally, we have an application in which we establish an existing result of an integral equation. We also provide illustrations of our results.

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Preliminaries

Definition 2.1 [12] The mapping $*:[0,1]\times[0,1]\longrightarrow[0,1]$ is said to be a t-norm when the following hold:

1. $*$ is commutative as well as associative,
2. $1*c_1=c_1$ whenever $c_1\in[0,1]$,
3. $c_1*c_2\leq c_3*c_4$ whenever $c_1\leq c_3$ and $c_2\leq c_4$, for each $c_1,c_2,c_3,c_4\in[0,1]$,
4. The operator $*$ is a continuous t-norm if $*$ is continuous.

Some illustrations of the above definition are given in [12].

Definition 2.2 [22] The 3-tuple $(A,G,*)$ called G-fuzzy metric space if A is any non-empty set, $*$ is a t-norm which is continuous and G is a fuzzy membership function on $A^3\times(0,\infty)$ which satisfies $z_1,z_2,z_3,z_4\in A$ and $t_1,t_2>0$:

1. $G(z_1,z_1,z_2,t_1)>0$,
2. $G(z_1,z_1,z_2,t_1)\geq G(z_1,z_2,z_3,t_1)$ with $z_2\neq z_3$,
3. $G(z_1,z_2,z_3,t_1)=1$ if and only if $z_1=z_2=z_3$,
4. $G(z_1,z_2,z_3,t_1)=G(p(z_1,z_2,z_3),t)$, where p is a permutation function,
5. $G(z_1,z_4,z_4,t_1)*G(z_4,z_2,z_3,t_2)\leq G(z_1,z_2,z_3,t_1+t_2)$ and
6. $G(z_1,z_2,z_3,.):(0,\infty)\longrightarrow[0,1]$ is continuous.

Example 2.3 [22] Let (A, G) be G-metric space. Let $c_1*c_2=c_1.c_2$ for all $c_1,c_2\in[0,1]$. For each $t_1>0$, $z_1,z_2\in A$, let

$$G(z_1,z_1,z_2,t_1)=\frac{t_1}{t_1+G(z_1,z_1,z_2)}.$$

Then $(A,G,*)$ is a G-fuzzy metric space.

Definition 2.4 [22] Let $(A,G,*)$ be a G-fuzzy metric space.

1. Any sequence $\{x_n\}$ in A converges to a point $z\in A$ if $\lim_{n\to\infty}G(x_n,x_n,z,s)=1$ for all $s>0$.
2. Any sequence $\{x_n\}$ in A is called a Cauchy sequence if corresponding to $0<\varepsilon<1$ and $s>0$, there is a positive integer n_0 for which $G(x_n,x_n,x_m,s)>1-\varepsilon$ when $n,m\geq n_0$.
3. If every Cauchy sequence converges, then the space is complete.

Lemma 2.5 [22] Let $(A,G,*)$ be a G-fuzzy metric space. Then $G(x_1,x_1,x_2,.)$ is nondecreasing for all $x_1,x_2\in A$.

Lemma 2.6 [22] G is a continuous function on $A^3\times(0,\infty)$.

Let A be a set with a partial order $\preceq$ and F be a function from A to itself. Then F is non-decreasing (non-increasing) whenever $x_1\preceq x_2$ $(x_1\succeq x_2)$ and $x_1,x_2\in A$ if $F(x_1)\preceq F(x_2)$ $(F(x_1)\succeq F(x_2))$ [3].

Definition 2.7 [3] Let $(A,\preceq)$ be a set with a partial ordering $\preceq$ and $F:A^2\to A$ be a function. The function F has the mixed monotone property whenever for all $x_1,x_2\in A$, $x_1\preceq x_2$ implies $F(x_1,y)\preceq F(x_2,y)$, with fixed $y\in A$ and, for all $y_1,y_2\in A$, $y_1\preceq y_2$ implies $F(x,y_1)\succeq F(x,y_2)$, with fixed $x\in A$.

Definition 2.8 [14] Let $(A,\preceq)$ be a partially ordered set and $F:A^2\to A$ and $g:A\to A$ be two functions. The function F has the mixed g-monotone property if for all $x_1,x_2\in A$, $g(x_1)\preceq g(x_2)$ implies $F(x_1,y)\preceq F(x_2,y)$, with any $y\in A$ and, for all $y_1,y_2\in A$, $g(y_1)\preceq g(y_2)$ implies $F(x,y_1)\succeq F(x,y_2)$, with fixed $x\in A$.

Definition 2.9 [3] Let A be any nonempty set. The ordered pair $(p,q)\in A\times A$ is a coupled fixed point of the function $F:A\times A\to A$ if

$F(p,q)=p$ as well as$F(q,p)=q$.

Definition 2.10 [14] Let A be any nonempty set. An element $(p,q)\in A^2$ is a coupled coincidence point of the functions $F:A\times A\to A$ and $g:A\to A$ if

$F(p,q)=g(p)$ and$F(q,p)=g(q)$.

Definition 2.11 [14] Let A be any non empty set. The functions $F:A\times A\to A$ and $g:A\to A$ are commuting if $(p,q)\in A^2$

$g(F(p,q))=F(g(p),g(q))$.

Definition 2.12 [20] Let (A, G) be a G-metric space. The pair (g, F) where $g:A\to A$ and $F:A\times A\to A$, is compatible if

$$\lim_{n\to\infty}G(g(F(p_n,q_n)),g(F(p_n,q_n)),F(g(p_n),g(q_n)))=0$$

and

$$\lim_{n\to\infty}G(g(F(q_n,p_n)),g(F(q_n,p_n)),F(g(q_n),g(p_n)))=0,$$

whenever $\{p_n\}$ and $\{q_n\}$ are sequences in A such that $\lim_{n\to\infty}F(p_n,q_n)=\lim_{n\to\infty}g(p_n)=p$ and $\lim_{n\to\infty}F(q_n,p_n)=\lim_{n\to\infty}g(q_n)=q$ for some $p,q\in A$.

The intuitive idea is that the functions F and g are commuting in the limit in the situations where the functional values are the same in the limit.

Definition 2.13 [13] Let $(A,G,*)$ be a G-fuzzy metric space. The pair (F, g) where $F:A\times A\to A$ and $g:A\to A$, are said to be compatible if for all $t>0$

$$\lim_{n\to\infty}G(g(F(p_n,q_n)),g(F(p_n,q_n)),F(g(p_n),g(q_n),t)=1$$

and

$$\lim_{n\to\infty} G(g(F(q_n,p_n)), g(F(q_n,p_n)), F(g(q_n), g(p_n), t) = 1,$$

whenever $\{p_n\}$ and $\{q_n\}$ are sequences in A such that $\lim_{n\to\infty} F(p_n,q_n) = \lim_{n\to\infty} g(p_n) = p$ and $\lim_{n\to\infty} F(q_n,p_n) = \lim_{n\to\infty} g(q_n) = q$ for some $p, q \in A$.

Lemma 2.14 Let (A, G) be a G-metric space. If the pair (F, g) where $F: A \times A \to A$ and $g: A \to A$ are compatible as per Definition 2.12, then he pair (F, g) is also compatible as per Definition 2.13.

Proof As we have mentioned earlier, in the associated G-fuzzy metric space, for all $x, y \in A$, $t > 0$,

$$G(x,x,y,t) = \frac{t}{t + G(x,x,y)} \tag{2.1}$$

and $a * b$ = minimum $\{a, b\}$.

Let $\{p_n\}$ and $\{q_n\}$ be two sequences in (X, G) such that $\lim_{n\to\infty} F(p_n,q_n) = \lim_{n\to\infty} g(p_n) = p$ and $\lim_{n\to\infty} F(q_n,p_n) = \lim_{n\to\infty} g(q_n) = q$. Then the same limits also hold in $(A, G, *)$.

Since g and F are compatible in (A, G), we have

$$\lim_{n\to\infty} G(g(F(p_n,q_n)), g(F(p_n,q_n)), F(g(p_n), g(q_n))) = 0$$

and

$$\lim_{n\to\infty} G(g(F(q_n,p_n)), g(F(q_n,p_n)), F(g(q_n), g(p_n))) = 0,$$

Now from (2.1), we have for all $t > 0$

$$G(g(F(p_n,q_n)), g(F(p_n,q_n)), F(g(p_n), g(q_n)), t) = \frac{t}{t + G(g(F(p_n,q_n)), g(F(p_n,q_n)), F(g(p_n), g(q_n)))}$$

and

$$G(g(F(q_n,p_n)), g(F(q_n,p_n)), F(g(q_n), g(p_n)), t) = \frac{t}{t + G(g(F(q_n,p_n)), g(F(q_n,p_n)), F(g(q_n), g(p_n)))}.$$

Taking $n \to \infty$ in both the above equalities, for all $t > 0$, we have

$$\lim_{n\to\infty} G(g(F(p_n,q_n)), g(F(p_n,q_n)), F(g(p_n), g(q_n)), t) = 1$$

and

$$\lim_{n\to\infty} G(g(F(q_n,p_n)), g(F(q_n,p_n)), F(g(q_n), g(p_n)), t) = 1.$$

Therefore (F, g) is compatible in $(A, G, *)$. □

Continuous *Hadžić* type t-norm is used in our theorem.

Definition 2.15 [12]*Hadžić* type t-norms are t-norms such that $\{*^p\}_{p \geq 0}$ give by

$*^0(s) = 1$, $*^{p+1}(s) = *(*^p(s), s)$ for all $p \geq 0$, $0 < s < 1$, are equi-continuous at $s = 1$, which is that, for $\epsilon > 0$ there exists $a(\epsilon) \in (0, 1)$ for which

$1 \geq u > a(\epsilon) \Rightarrow *^p(u) > 1 - \epsilon$ for all $p \geq 0$.

Illustrations of the above t-norm type is given in [12].

Lemma 2.16 Let $(A, G, *)$ be a G-fuzzy metric space having a t-norm of *Hadžić* type for which $G(x, x, y, u) \to 1$ as $u \to \infty$, $\{x_n\}$ and $\{y_n\}$ in A satisfy, for all $n \geq 1$, $t > 0$,

$$G(x_n, x_n, x_{n+1}, t) * G(y_n, y_n, y_{n+1}, t) \geq G\Big(x_{n-1}, x_{n-1}, x_n, \frac{t}{k}\Big) * G\Big(y_{n-1}, y_{n-1}, y_n, \frac{t}{k}\Big) \tag{2.2}$$

with some $0 < k < 1$, then $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences.

Proof We successively apply (2.2) to obtain for all $i \geq 1$ (integer) $t > 0$, $q \geq 0$,

$$G(x_{q+i}, x_{q+i}, x_{q+i+1}, t) * G(y_{q+i}, y_{q+i}, y_{q+i+1}, t) \geq G\Big(x_q, x_q, x_{q+1}, \frac{t}{k^i}\Big) * G\Big(y_q, y_q, y_{q+1}, \frac{t}{k^i}\Big) \tag{2.3}$$

Let $\epsilon > 0$ and $0 < \lambda < 1$ be given. Let p be another integer such that $p > q$ be some other integer. Then

$$\epsilon = \epsilon\frac{(1-k)}{(1-k)} > \epsilon(1-k)(1 + k + \cdots + k^{p-q-1}).$$

Then, by Lemma 2.5, for all $p > q$, we obtain

$$G(x_q, x_q, x_p, \epsilon) * G(y_q, y_q, y_p, \epsilon) \geq G(x_q, x_q, x_p, \epsilon(1-k)(1 + k + \cdots + k^{p-q-1})) * G(y_q, y_q, y_p, \epsilon(1-k)(1 + k + \cdots + k^{p-q-1})),$$

or,

$$\begin{aligned} G(x_q, x_q, x_p, \epsilon) * G(y_q, y_q, y_p, \epsilon) &\geq G(x_q, x_q, x_{q+1}, \epsilon(1-k)) * G(x_{q+1}, x_{q+1}, x_{q+2}, \epsilon k(1-k)) * \cdots * \\ &\quad G(x_{p-1}, x_{p-1}, x_p, \epsilon k^{p-q-1}(1-k)) * G(y_q, y_q, y_{q+1}, \epsilon(1-k)) \\ &\quad * G(y_{q+1}, y_{q+1}, y_{q+2}, \epsilon k(1-k)) * \cdots * G(y_{p-1}, y_{p-1}, y_p, \epsilon k^{p-q-1}(1-k)). \\ &= \{G(x_q, x_q, x_{q+1}, \epsilon(1-k)) * G(y_q, y_q, y_{q+1}, \epsilon(1-k))\} \\ &\quad * \{G(x_{q+1}, x_{q+1}, x_{q+2}, \epsilon k(1-k)) * G(y_{q+1}, y_{q+1}, y_{q+2}, \epsilon k(1-k))\} * \cdots * \\ &\quad \{G(x_{p-1}, x_{p-1}, x_p, \epsilon k^{p-q-1}(1-k)) * G(y_{p-1}, y_{p-1}, y_p, \epsilon k^{p-q-1}(1-k))\}. \end{aligned} \tag{2.4}$$

We put $t = (1-k)\epsilon k^i$ in (2.3); we get, for all $q \geq 0, i \geq 1$

$$G(x_{q+i}, x_{q+i}, x_{q+i+1}, (1-k)\epsilon k^i) * G(y_{q+i}, y_{q+i}, y_{q+i+1}, (1-k)\epsilon k^i) \geq G(x_q, x_q, x_{q+1}, (1-k)\epsilon) * G(y_q, y_q, y_{q+1}, (1-k)\epsilon).$$

From the above, and using (2.4), with $p > q$, we get

$$G(x_q, x_q, x_p, \epsilon) * G(y_q, y_q, y_p, \epsilon) \geq \{G(x_q, x_q, x_{q+1}, \epsilon(1-k)) * G(y_q, y_q, y_{q+1}, \epsilon(1-k))\} * \{G(x_{q+1}, x_{q+1}, x_{q+2}, \epsilon(1-k)) * G(y_{q+1}, y_{q+1}, y_{q+2}, \epsilon(1-k))\} * \cdots * \{G(x_{p-1}, x_{p-1}, x_p, \epsilon(1-k)) * G(y_{p-1}, y_{p-1}, y_p, \epsilon(1-k))\},$$

that is, $G(x_q, x_q, x_p, \epsilon) * G(y_q, y_q, y_p, \epsilon)$

$$\geq *^{(p-q)} \{G(x_q, x_q, x_{q+1}, \epsilon(1-k)) * G(y_q, y_q, y_{q+1}, \epsilon(1-k))\}. \tag{2.5}$$

By equi-continuity of t-norm at 1, there exists $\eta(\lambda) \in (0, 1)$ such that for all $m > n$,

$$*^{(m-n)}(s) > 1 - \lambda, \tag{2.6}$$

whenever $1 \geq s > \eta(\lambda)$, where $0 < \lambda < 1$, as mentioned above, is given. Since $G(x_0, x_0, x_1, u) \to 1$ as $u \to \infty$, there exists $N(\epsilon, \lambda)$ such that

$$G(x_0, x_0, x_1, \frac{(1-k)\epsilon}{k^n}) * G(y_0, y_0, y_1, \frac{(1-k)\epsilon}{k^n}) > \eta(\lambda) \text{whenever} n \geq N(\epsilon, \lambda). \tag{2.7}$$

From (2.3) and (2.7), with $q = 0, i = n \geq N(\epsilon, \lambda)$ and $t = (1-k)\epsilon$, we have

$$G(x_n, x_n, x_{n+1}, (1-k)\epsilon) * G(y_n, y_n, y_{n+1}, (1-k)\epsilon) > \eta(\lambda).$$

Then, from (2.6), with $s = G(x_n, x_n, x_{n+1}, (1-k)\epsilon) * G(y_n, y_n, y_{n+1}, (1-k)\epsilon)$ and $m > n \geq N(\epsilon, \lambda)$, we have

$$*^{(m-n)} (G(x_n, x_n, x_{n+1}, (1-k)\epsilon) * G(y_n, y_n, y_{n+1}, (1-k)\epsilon)) > 1 - \lambda.$$

Then, by (2.5), for all $m > n \geq N(\epsilon, \lambda)$, we obtain

$$G(x_n, x_n, x_m, \epsilon) * G(y_n, y_n, y_m, \epsilon) > 1 - \lambda,$$

which implies that

$$G(x_n, x_n, x_m, \epsilon) > 1 - \lambda \text{and} G(y_n, y_n, y_m, \epsilon) > 1 - \lambda \text{ for all } n, m \geq N(\epsilon, \lambda).$$

Again $\epsilon > 0$ and λ are arbitrary with their range.

This proves that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. □

Note 2.17 Equi-continuous of the iterates is essential in the proof of the above lemma.

Main results

Theorem 3.1 Let $(A, G, *)$ be a complete G-fuzzy metric space having a t-norm whose *Hadžić* type is such that $G(x, y, z, s) \to 1$ as $s \to \infty$, for all $x, y, z \in A$. Let $\preceq$ be a partial ordering on A. Let $F : A \times A \to A$ and $g : A \to A$ be two functions of which F has mixed g-monotone property and that the following is satisfied:

$$G(F(x,y), F(x,y), F(u,v), ks) * G(F(y,x), F(y,x), F(v,u), ks) \geq G(g(x), g(x), g(u), s) * G(g(y), g(y), g(v), s),$$

for all $x, y, u, v \in A$, $s > 0$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$, where $0 < k < 1$ and $F(A \times A) \subseteq g(A)$, g is continuous and monotonic increasing, (g, F) is a compatible pair. Suppose one of (a) and (b) holds:

(a) F is continuous
(b)

$$1. \{z_p\} \to z \text{ is such that } z_p \preceq z_{p+1}, \text{for every } p \geq 0, \text{ then } z_p \preceq z \text{for every } p \geq 0, \tag{3.2}$$

$$2. \{z_p\} \to z \text{ is such that } z_p \succeq z_{p+1} \text{for every } p \geq 0, \text{then } z_p \succeq z \text{for every } p \geq 0 \tag{3.3}$$

If there are $x_0, y_0 \in A$ for which $g(x_0) \preceq F(x_0, y_0)$, $g(y_0) \succeq F(y_0, x_0)$, then we can find $x, y \in A$ for which $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Proof Let x_0, y_0 in A, for which $g(x_0) \preceq F(x_0, y_0)$, $g(y_0) \succeq F(y_0, x_0)$. We construct the sequences $\{x_p\}$ and $\{y_p\}$ in A, for all $p \geq 0$,

$$g(x_{p+1}) = F(x_p, y_p) \text{ and} g(y_{p+1}) = F(y_p, x_p). \tag{3.4}$$

Then it follows that for all $p \geq 0$

$$g(x_p) \preceq g(x_{p+1}) \tag{3.5}$$

and

$$g(y_p) \succeq g(y_{p+1}). \tag{3.6}$$

Let for all $s > 0$, $p \geq 0$, due to (3.4), (3.5) and (3.6), from (3.1), for all $s > 0$, $k \geq 1$, we have

$$G(g(x_p), g(x_p), g(x_{p+1}), ks) * G(g(y_p), g(y_p), g(y_{p+1}), ks) = G(F(x_{p-1}, y_{p-1}), F(x_{p-1}, y_{p-1}), F(x_p, y_p), ks) * G(F(y_{p-1}, x_{p-1}), F(y_{p-1}, x_{p-1}), F(y_p, x_p), ks) \geq G(g(x_{p-1}), g(x_{p-1}), g(x_p), s) * G(g(y_{p-1}), g(y_{p-1}), g(y_p), s)$$

that is,

$$G(g(x_p), g(x_p), g(x_{p+1}), s) * G(g(y_p), g(y_p), g(y_{p+1}), s) \geq G\Big(g(x_{p-1}), g(x_{p-1}), g(x_p), \frac{s}{k}\Big) * G\Big(g(y_{p-1}), g(y_{p-1}), g(y_p), \frac{s}{k}\Big). \tag{3.7}$$

From (3.7), by the Lemma 2.16, we conclude that $\{g(x_p)\}$ and $\{g(y_p)\}$ are Cauchy sequences. Since A is complete, there exist $x, y \in A$ such that

$$\lim_{p\to\infty} g(x_p) = x \text{ and } \lim_{n\to\infty} g(y_p) = y \tag{3.8}$$

Therefore,
$\lim_{p\to\infty} g(x_{p+1}) = \lim_{p\to\infty} F(x_p, y_p) = x$ and $\lim_{p\to\infty} g(y_{p+1}) = \lim_{p\to\infty} F(y_p, x_p) = y$.

Since (g, F) is a compatible pair, using continuity of g and Definition 2.13, we have

$$g(x) = \lim_{p\to\infty} g(g(x_{p+1})) = \lim_{p\to\infty} g(F(x_p, y_p)) = \lim_{p\to\infty} F(g(x_p), g(y_p)) \tag{3.9}$$

and

$$g(y) = \lim_{p\to\infty} g(g(y_{p+1})) = \lim_{p\to\infty} g(F(y_p, x_p)) = \lim_{p\to\infty} F(g(y_p), g(x_p)). \tag{3.10}$$

Now assume that(a) holds.

Then from (3.9), (3.10) and by using (3.8), we have

$$g(x) = \lim_{p\to\infty} g(F(x_p, y_p)) = \lim_{p\to\infty} F(g(x_p), g(y_p)) = F(\lim_{p\to\infty} g(x_p), \lim_{p\to\infty} g(y_p)) = F(x, y)$$

and

$$g(y) = \lim_{p\to\infty} g(F(y_p, x_p)) = \lim_{p\to\infty} F(g(y_p), g(x_p)) = F(\lim_{p\to\infty} g(y_p), \lim_{p\to\infty} g(x_p)) = F(y, x).$$

therefore $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Next we assume that (b) holds.

By (3.5), (3.6) and (3.8), it follows that, for all $n \geq 0$,

$g(x_p) \preceq x$ and $g(y_p) \succeq y$.

Since g is monotonic increasing,

$$g(g(x_p)) \preceq g(x) \text{ and } g(g(y_p)) \succeq g(y). \tag{3.11}$$

Now, for all $s > 0$, $p \geq 0$, we have

$$G(F(x, y), F(x, y), g(F(x_p, y_p)), s) \geq G(F(x, y), F(x, y), g(g(x_{p+1})), ks) G(g(g(x_{p+1})), g(g(x_{p+1})), g(F(x_p, y_p)), (s - ks)).$$

Taking $p \to \infty$ on the both sides of the above inequality, for all $s > 0$,

$$\lim_{p\to\infty} G(F(x, y), F(x, y), g(F(x_p, y_p)), s) \geq \lim_{p\to\infty} [G(F(x, y), F(x, y), g(g(x_{p+1})), ks) G(g(g(x_{p+1})), g(g(x_{p+1})), g(F(x_p, y_p)), (s - ks))],$$

that is, $G(F(x, y), F(x, y), g(x), s)$

$$\begin{aligned}
&= \lim_{p\to\infty} [G(F(x, y), F(x, y), g(F(x_p, y_p)), ks) * G(g(g(x_{p+1})), g(g(x_{p+1})), g(x), (s - ks))] \\
&= G(F(x, y), F(x, y), \lim_{p\to\infty} g(F(x_p, y_p)), ks) * G(\lim_{p\to\infty} g(g(x_{p+1})), \lim_{p\to\infty} g(g(x_{p+1})), g(x), (s - ks))(\text{by lemma 2.6}) \\
&= G(F(x, y), F(x, y), \lim_{p\to\infty} F(g(x_p), g(y_p)), ks) * G(g(x), g(x), g(x), (s - ks)(\text{by 3.9}) \\
&= \lim_{p\to\infty} G(F(g(x_p), g(y_p)), F(x, y), F(x, y), ks) * 1(\text{by lemma 2.6}) \\
&= \lim_{p\to\infty} G(F(g(x_p), g(y_p)), F(x, y), F(x, y), ks),
\end{aligned}$$

that is, $G(F(x, y), F(x, y), g(x), s) \geq \lim_{p\to\infty} G(F(g(x_p), g(y_p)), F(x, y), F(x, y), ks)$.Similarly we obtain for all $s > 0$

$$G(F(y, x), F(y, x), g(y), s) \geq \lim_{p\to\infty} G(F(g(y_p), g(x_p)), F(y, x), F(y, x), ks). \tag{3.13}$$

From (3.12) and (3.13), using (3.1) and (3.11), for all $s > 0$, we have

$$\begin{aligned}
&G(F(x, y), F(x, y), g(x), s) * G(F(y, x), F(y, x), g(y), s) \\
&\geq \lim_{p\to\infty} [G(F(g(x_p), g(y_p)), F(x, y), F(x, y), ks) * G(F(g(y_p), g(x_p)), F(y, x), F(y, x), ks)] \\
&\geq \lim_{p\to\infty} [G(g(g(x_p)), g(x), g(x), s) * G(g(g(y_p)), g(y), g(y), s)](since * is continuous) = G(\lim_{p\to\infty} g(g(x_p)), g(x), g(x), s) * G(\lim_{p\to\infty} g(g(y_p)), g(y), g(y), s) \\
&= G(g(x), g(x), g(x), s) * G(g(y), g(y), g(y), s)(\text{by}(3.9)) \\
&= 1 * 1 = 1,
\end{aligned}$$

that is,
$G(F(x, y), g(x), g(x), s) * G(F(y, x), g(y), g(y), s) \geq 1$, which implies that $g(x) = F(x, y)$ and $g(y) = F(y, x)$. Hence the proof.

Corollary 3.2 Let $(A, G, *)$ be a complete G-fuzzy metric space having a t-norm which is *Hadžić* type for which $G(x, y, z, s) \to 1$ as $s \to \infty$, for all $x, y, z \in A$. Let $\preceq$ be a partial ordering on A. Let $F : A \times A \to A$ and $g : A \to A$ be two functions out of which F has mixed g-monotone property and satisfies the following condition:

$$G(F(x,y),F(x,y),F(u,v),ks) \\ *G(F(y,x),F(y,x),F(v,u),ks) \geq G(g(x),g(x),g(u),s) \\ *G(g(y),g(y),g(v)s),$$

for all $x,y,u,v \in A$, $s > 0$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$, where $0<k<1$ and $F(A \times A) \subseteq g(A)$, g is continuous and monotonic increasing, (g, F) is a commuting pair. Suppose either

(a) F is continuous or
(b) (3.2) and (3.3) hold.

If there are $x_0, y_0 \in A$ for which $g(x_0) \preceq F(x_0,y_0)$, $g(y_0) \succeq F(y_0,x_0)$, then we obtain $x,y \in A$ satisfying $g(x) = F(x,y)$ and $g(y) = F(y,x)$.

Proof As commuting pairs are also a compatible pairs, the result follows from Theorem 3.1.

Later, through an example, it is established that the Corollary 3.2 is actually contained within Theorem 3.1. □

Corollary 3.3 Let $(A,G,*)$ be a complete G-fuzzy metric space having a t-norm which is *Hadžić* type for which $G(x,y,z,s) \to 1$ as $s \to \infty$, for all $x,y,z \in A$. Let $\preceq$ be a partial ordering on A. Let $F : A \times A \to A$ be a function for which F has mixed mixed monotone property and satisfies the following condition:

$$G(F(x,y),F(x,y),F(u,v),ks) \\ *G(F(y,x),F(y,x),F(v,u),ks) \geq G(x,x,u,s) \\ *G(y,y,v,s),$$

for all $x,y,u,v \in A$, $s > 0$ such that $x \preceq u$ and $y \succeq v$, and $0<k<1$ and $F(A \times A) \subseteq A$. Suppose either

(a) F is continuous or
(b) (3.2) and (3.3) hold.

If there are $x_0, y_0 \in A$ for which $x_0 \preceq F(x_0,y_0)$, $y_0 \succeq F(y_0,x_0)$, then we obtain $x,y \in A$ such that $x = F(x,y)$ and $y = F(y,x)$

Proof With the assumption of $g = I$, the corollary follows by an application of Theorem 3.1. □

Example 3.4 Let $(A,\preceq)$ be a partially ordered set with $A = [0,1]$ and the usual relation ordering $\leq$ on real numbers be the partial ordering $\preceq$. Let for all $s > 0$, $p,q,z \in A$,

$$G(p,q,z,s) = e^{-\dfrac{|p-q|+|q-z|+|z-p|}{s}}.$$

Let $a*b = \min\{a,b\}$. Then $(A,G,*)$ is a complete G-fuzzy metric space such that $G(p,q,z,s) \to 1$ as $s \to \infty$, for all $p,q \in A$.

Let the mapping $g : A \to A$ be defined as

$$g(p) = \frac{5}{6}p^2 \text{ for all } p \in A$$

and the mapping $F : A \times A \to A$ be defined as

$$F(p,q) = \frac{p^2-q^2}{4}.$$

Then $F(A \times A) \subseteq g(A)$ and F satisfies the mixed g-monotone property.

Let $\{t_n\}$ and $\{r_n\}$ be sequences in A such that

$$\lim_{n\to\infty} F(t_n,r_n) = a, \quad \lim_{n\to\infty} g(t_n) = a, \lim_{n\to\infty} F(r_n,t_n) \\ = b \text{ and } \lim_{n\to\infty} g(r_n) = b.$$

Now, for all $n \geq 0$,

$$g(t_n) = \frac{5}{6}t_n^2, \; g(r_n) = \frac{5}{6}r_n^2, F(t_n,r_n) = \frac{t_n^2-r_n^2}{4}$$

and

$$F(r_n,t_n) = \frac{r_n^2-t_n^2}{4}.$$

Then necessarily $a = 0$ and $b = 0$.

It then follows from Lemma 2.6 that, for all $s > 0$,

$$\lim_{n\to\infty} G(g(F(t_n,r_n)),g(F(t_n,r_n)),F(g(t_n),g(r_n),s) = 1$$

and

$$\lim_{n\to\infty} G(g(F(r_n,t_n)),g(F(r_n,t_n)),F(g(r_n),g(t_n)),s) = 1.$$

Therefore, (g, F) is compatible pair in A. Now we show that the inequality (3.1) holds.

$$2|F(p,q)-F(u,v)| \leq |g(p)-g(u)| + |g(q) \\ -g(v)|, p \geq u, q \leq v \tag{3.14}$$

and

$$2|F(q,p)-F(v,u)| \leq |g(q)-g(v)| + |g(p) \\ -g(u)|, p \geq u, q \leq v. \tag{3.15}$$

From (3.14), for all $s > 0$ and $0<k<1$, we have

$$e^{-\frac{2|F(p,q)-F(u,v)|}{ks}} \geq e^{-\frac{[|g(p)-g(u)|+|g(q)-g(v)|]}{s}} \geq e^{-\frac{2|g(p)-g(u)|}{2s}}.e^{-\frac{2|g(q)-g(v)|}{2s}} \\ \geq \sqrt{e^{-\frac{2|g(p)-g(u)|}{s}}.e^{-\frac{2|g(q)-g(v)|}{s}}} \geq \min\left\{e^{-\frac{2|g(p)-g(u)|}{s}}e^{-\frac{2|g(q)-g(v)|}{s}}\right\} \\ e^{-\frac{2|F(p,q)-F(u,v)|}{ks}} \geq \min\{G(g(p),g(p),g(u),s),M(g(q), \\ g(q),g(v),s)\} \tag{3.16}$$

Similarly from (3.15), we get

$$e^{-\frac{2|F(q,p)-F(v,u)|}{ks}} \geq \min\{G(g(p),g(p),g(u),s),M(g(q), \\ g(q),g(v),s)\}. \tag{3.17}$$

From (3.16) and (3.17), we obtain

$$\min\{G(F(p,q),F(p,q),F(u,v),ks),G(F(q,p),F(q,p),F(v,u),ks)\} \geq \min\{G(g(p),g(p),g(u),s),G(g(q),g(q),g(v),s)\}, \text{that is}, G(F(p,q),F(p,q),F(u,v),ks) * G(F(q,p),F(q,p),F(v,u),ks) \geq G(g(p),g(p),g(u),s) * G(g(q),g(q),g(v),s).$$

Hence (3.1) holds. Other cases can be similarly done. Then Theorem 3.1 is applicable. As can be seen by observation (0, 0) is the coupled coincidence point for the pair (g, F).

Remark 3.5 (g, F) is not a commuting pair in Example 3.4 for which Corollary 3.2 is not applicable to this example. This establishes that Theorem 3.1 is actually more general than its Corollary 3.2.

Results in G-metric spaces

Here we apply Theorem 3.1 to obtain a coupled coincidence point results in G-metric spaces. Several existing theorems are hereby extended.

Theorem 4.1 Let A be a non empty set with a partial order $\preceq$ and G be a G- metric on A such that (A, G) is a complete G- metric space. Let $F : A \times A \to A$ and $g : A \to A$ be two mappings such that F has the mixed g-monotone property and satisfies the following condition:

$$\max\{G(F(p,q),F(p,q),F(u,v)),G(F(q,p),F(q,p),F(v,u))\} \leq \frac{k}{2}[G(g(p),g(p),g(u))+G(g(q),g(q),g(v))], \tag{4.1}$$

with $p,q,u,v \in A$ such that $g(p) \preceq g(u)$ and $g(q) \succeq g(v)$, and for some fixed $k \in (0,1)$. Suppose $F(A \times A) \subseteq g(A)$, ($g$, F) is compatible pair of functions in which we assume the continuity of g. Further, either

(a) F is continuous or
(b) (3.2) and (3.3) hold.

If there are $x_0, y_0 \in A$ such that $g(x_0) \preceq F(x_0, y_0)$, $g(y_0) \succeq F(y_0, x_0)$, then the pair ($g$, F) has a coupled coincidence point in A.

Proof For all $p,q,z \in A$ and $s > 0$, we define

$$G(p,q,z,s) = \frac{s}{s+G(p,q,z)}$$

and $a * b = \min\{a,b\}$. Then, $(A,G,*)$ is a complete G-fuzzy metric space. Further, from the form of G, $G(p,q,z,s) \to 1$ as $s \to \infty$, whenever $p,q,z \in A$. Using Lemma 2.14, we conclude that (g, F) is a compatible pair in this G-fuzzy metric space. Next we show that the inequality (4.1) implies (3.1). If otherwise, from (3.1), for some $s > 0$, $p,q,u,v \in A$ with $g(p) \preceq g(u)$ and $g(q) \succeq g(v)$, we have

$$\min\left\{\frac{s}{s+\frac{1}{k}G(F(p,q),F(p,q),F(u,v))},\frac{s}{s+\frac{1}{k}G(F(q,p),F(q,p),F(v,u))}\right\} < \min\left\{\frac{s}{s+G(g(p),g(p),g(u))},\frac{s}{s+G(g(q),g(q),g(v))}\right\},$$

Form the above inequality, we have either

$$\frac{s}{s+\frac{1}{k}G(F(p,q),F(p,q),F(u,v))} < \min\left\{\frac{s}{s+G(g(p),g(p),g(u))},\frac{s}{s+G(g(q),g(q),g(v))}\right\} \tag{4.2}$$

or

$$\frac{s}{s+\frac{1}{k}G(F(q,p),F(q,p),F(v,u))}\} < \min\left\{\frac{s}{s+G(g(p),g(p),g(u))},\frac{s}{s+G(g(q),g(q),g(v))}\right\}. \tag{4.3}$$

From (4.2), we have

$$s+\frac{1}{k}G(F(p,q),F(p,q),F(u,v)) > s + G(g(p),g(p),g(u)) \text{ and} s + \frac{1}{k}G(F(p,q),F(p,q),F(u,v)) > s + G(g(q),g(q),g(v)).$$

Combining the above two inequalities, we have that

$$G(F(p,q),F(p,q),F(u,v)) > \frac{k}{2}[G(g(p),g(p),g(u)) + G(g(q),g(q),g(v))]. \tag{4.4}$$

Similarly from (4.3), we have

$$G(F(q,p),F(q,p),F(v,u)) > \frac{k}{2}[G(g(q),g(q),g(v)) + G(g(p),g(p),g(u))]. \tag{4.5}$$

By (4.4) and (4.5), we have

$$\max\{G(F(p,q),F(p,q),F(u,v)),G(F(q,p),F(q,p),F(v,u))\} > \frac{k}{2}[G(g(p),g(p),g(u))+G(g(q),g(q),g(v))],$$

which contradicts (4.1). The proof is then completed by an application of Theorem 3.1. □

Corollary 4.2 Let $(A,\preceq)$ be any non empty set with partial order $\preceq$ and G be a G-metric on A for which (A, G) is a complete G-metric space. Let $F : A \times A \to A$

and $g : A \to A$ be two mappings such that F satisfies the property given in definition 2.8 and the following inequality:

$$\begin{aligned}&[G(F(p,q),F(p,q),F(u,v))\\&\quad+G(F(q,p),F(q,p),F(v,u))]\leq k[G(g(p),g(p),g(u))\\&\quad+G(g(q),g(q),g(v))],\end{aligned} \tag{4.6}$$

with $p,q,u,v \in A$ such that $g(p) \preceq g(u)$ and $g(q) \succeq g(v)$ and some fixed $k \in (0,1)$. Suppose $F(A \times A) \subseteq g(A)$, ($g$, F) is compatible pair of functions in which we assume the continuity of g. Also suppose either

(a) F is continuous or
(b) (3.2) and (3.3) hold.

If there are $x_0, y_0 \in A$ such that $g(x_0) \preceq F(x_0,y_0)$, $g(y_0) \succeq F(y_0,x_0)$, then the pair ($g$, F) has a coupled coincidence point in A.

Proof Since $\frac{x+y}{2} \leq \max\{x,y\}$, the proof follows from Theorem 4.1. □

Example 4.3 Let $A=[0,1]$ and with the natural ordering $\leq$ in $\mathbb{R}$ as the partial ordering $\preceq$. Let $p,q,z \in A, G(p,q,z)=|p-q|+|q-z|+|z-p|$. Then ($A$, G) is a complete G-metric space. Let the mapping $g: A \to A$ be given by

$$g(p)=\frac{5}{6}p^2, p \in A$$

and the mapping $F : A^2 \to A$ be defined as

$$F(p,q)=\frac{p^2-2q^2}{4}.$$

Then $F(A \times A) \subseteq g(A)$ and F satisfies the property in Definition 2.8. Let $\{t_n\}$ and $\{r_n\}$ be two sequences in A for which

$$\lim_{n\to\infty}F(t_n,r_n)=a, \quad \lim_{n\to\infty}g(t_n)=a, \lim_{n\to\infty}F(r_n,t_n) = b \text{ and } \lim_{n\to\infty}g(r_n)=b.$$

Now, for all $n \geq 0$,

$$g(t_n)=\frac{5}{6}t_n^2, \quad g(r_n)=\frac{5}{6}r_n^2, F(t_n,r_n)=\frac{t_n^2-r_n^2}{4}$$

and

$$F(r_n,t_n)=\frac{r_n^2-t_n^2}{4}.$$

Then necessarily $a=0$ and $b=0$. Applying Lemma 2.6, we have

$$\lim_{n\to\infty}G(g(F(t_n,r_n)),g(F(t_n,r_n)),F(g(t_n),g(r_n))=0$$

and

$$\lim_{n\to\infty}G(g(F(r_n,t_n)),g(F(r_n,t_n)),F(g(r_n),g(t_n)))=0.$$

Therefore, the mappings (g, F) are compatible pair in A. The mappings are not commuting. Now we show that the condition (4.6) holds.

$$\begin{aligned}2|F(p,q)-F(u,v)| &\leq \frac{3}{10}\cdot 2|g(p)-g(u)|\\&+\frac{3}{5}\cdot 2|g(q)-g(v)|, p \geq u, q \leq v\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}2|F(q,p)-F(v,u)| &\leq \frac{3}{10}\cdot 2|g(q)-g(v)|\\&+\frac{3}{5}\cdot 2|g(p)-g(u)|, p \geq u, q \leq v.\end{aligned} \tag{4.8}$$

Adding (4.7) and (4.8), we get the inequality (4.6) with $k=\frac{9}{10}$. Other cases can be similarity shown. Here (0, 0) is a coupled coincidence point.

Application to integral equations

Here give an application of a result in the previous section to a problem of an integral equation. Similar applications to integral equation problem results have been made in [2, 4, 21]. We consider the integral equation

$$\begin{aligned}p(t) = &\int_a^t (m_1(t,s)+m_2(t,s))(f_1(s,p(s))+f_2(s,p(s)))ds\\&+h(t), t \in [a,\infty),\end{aligned} \tag{5.1}$$

where $h \in L[a,\infty)$, $m_1(t,s), m_2(t,s), f_1(s,p(s)), f_2(s,p(s))$ are real-valued functions that are measurable both in t and s.

Assumption 5.1 The functions $m_1(t,s), m_2(t,s), f_1(s,p(s)), f_2(s,p(s))$ satisfy the following:

(1) $m_1(t,s) \geq 0$, $t,s \in [a,\infty)$ and $\int_a^t sup_{s\in[0,\infty)}|m_1(t,s)| dt = \frac{M_1}{2} < \infty$,
(2) $m_2(t,s) \leq 0$, $t,s \in [a,\infty)$ and $\int_a^t sup_{s\in[0,\infty)}|m_2(t,s)| dt = \frac{M_2}{2} < \infty$,
(3) $f_1(s,p(s)), f_2(s,p(s) \in L[a,\infty)$ for all $p \in L[a,\infty)$ and there exist $\lambda, \mu > 0$ such that

$$\begin{aligned}&0 \leq f_1(s,p(s))-f_1(s,q(s)) \leq \lambda(p(s)-q(s)) \text{ and}\\&-\mu(p(s)-q(s)) \leq f_2(s,p(s))-f_2(s,q(s)) \leq 0\end{aligned}$$

for all $p,q \in L[a,\infty)$ with $q(s)b \leq p(s)$.

Definition 5.2 An element $(c,w) \in L[a,\infty) \times L[a,\infty)$ is called a coupled lower and upper solution of the integral equation (5.1) if $c(t) \leq w(t)$ and

$$\begin{aligned}c(t) &\le \int_a^t (m_1(t,s)(f_1(s,c(s)) + f_2(s,w(s)))ds \\ &+ \int_a^t (m_2(t,s)(f_1(s,c(s)) + f_2(s,w(s)))ds \\ &+ h(t), w(t) \le \int_a^t (m_1(t,s)(f_1(s,w(s)) + f_2(s,c(s)))ds \\ &+ \int_a^t (m_2(t,s)(f_1(s,w(s)) + f_2(s,c(s)))ds \\ &+ h(t),\ \text{where } t \\ &\in L[a,\infty).\end{aligned}$$

Theorem 5.3 With the Assumption 5.1, if (5.1) has a coupled lower and upper solution and $\frac{(\lambda+\mu)(M_1+M_2)}{2} < 1$, then it has a solution in $L[a,\infty)$.

Proof We consider the complete G-metric space (A, G) where $A = L[a,\infty)$ and $G(p,q,r) = \int_a^\infty [|p(t) - q(t)| + |q(t) - r(t)| + |r(t) - p(t)|]dt$, for all $p,q,r \in A$. For every $p,q \in A$, let

$$\begin{aligned}F(p,q)(t) = &\int_a^t (m_1(t,s)(f_1(s,p(s)) + f_2(s,q(s)))ds \\ &+ \int_a^t m_2(t,s)(f_1(s,p(s)) + f_2(s,q(s)))ds \\ &+ h(t), t \in [a,\infty).\end{aligned}$$

It follows from Theorem 3.2 of [15] that $F(p, q)$ has the mixed monotone property. Now for $p,q,u,v \in X$, $p \ge q$, $u \le v$, we have

$$\begin{aligned}&F(p,q)(t) - F(u,v)(t) \\ &= \Big| \int_a^t (m_1(t,s)(f_1(s,p(s)) + f_2(s,q(s)))ds \\ &\quad + \int_a^t m_2(t,s)(f_1(s,q(s)) + f_2(s,p(s)))ds \\ &\quad - \Big(\int_a^t (m_1(t,s)(f_1(s,u(s)) + f_2(s,v(s)))ds \\ &\quad + \int_a^t m_2(t,s)(f_1(s,v(s)) + f_2(s,u(s)))ds\Big)\Big|, \\ &= \Big| \int_a^t (m_1(t,s)[(f_1(s,p(s)) - (f_1(s,u(s))) \\ &\quad - (f_2(s,v(s)) - f_2(s,q(s)))]ds \\ &\quad + \int_a^t m_2(t,s)[(f_1(s,v(s)) - f_1(s,q(s))) \\ &\quad - (f_2(s,p(s)) - f_2(s,u(s)))]ds\Big|, \le \Big| \int_a^t (m_1(t,s)[\lambda(p(s) \\ &\quad - u(s)) + \mu(q(s) - v(s))]ds + \int_a^t m_2(t,s)[(\lambda(v(s) \\ &\quad - q(s)) + \mu(p(s) - u(s))]ds\Big|.\end{aligned}$$

Since $m_1(t,s) \ge 0$, $m_2(t,s) \le 0$, $p(s) - u(s) \ge 0$, $v(s) - q(s) \ge 0$ and $\lambda, \mu \ge 0$, we have

$$\begin{aligned}&2\big|F(p,q)(t) - F(u,v)(t)\big| \le 2\Big(\int_a^t (m_1(t,s)[\lambda(p(s) - u(s)) \\ &\quad + \mu(q(s) - v(s))]ds + \int_a^t m_2(t,s)[(\lambda(v(s) - q(s)) \\ &\quad + \mu(p(s) - u(s))]ds\Big) \le sup_{s\in[0,\infty)}|m_1(t,s)| \int_a^t [\lambda 2|(p(s) \\ &\quad - u(s))| + \mu 2|(q(s) - v(s))|]ds \\ &\quad + sup_{s\in[0,\infty)}|m_2(t,s) \int_a^t m_2(t,s)[(\lambda 2|(v(s) - q(s))| \\ &\quad + \mu 2|(p(s) - u(s))|]ds \le sup_{s\in[0,\infty)}|m_1(t,s)| \int_a^\infty [\lambda|(p(s) \\ &\quad - u(s))| + \mu|(q(s) - v(s))|]ds \\ &\quad + sup_{s\in[0,\infty)}|m_2(t,s) \int_a^\infty m_2(t,s)[(\lambda 2|(v(s) - q(s))| \\ &\quad + \mu 2|(p(s) - u(s))|]ds.\end{aligned}$$

Therefore,

$$\begin{aligned}&2|F(p,q)(t) \\ &\quad - F(u,v)(t)| \le sup_{s\in[0,\infty)}|m_1(t,s)| \int_a^\infty [\lambda 2|(p(s) - u(s))| \\ &\quad + \mu 2|(q(s) - v(s))|]ds \\ &\quad + sup_{s\in[0,\infty)}|m_2(t,s) \int_a^\infty m_2(t,s)[(\lambda 2|(v(s) - q(s))| \\ &\quad + \mu 2|(p(s) \\ &\quad - u(s))|]ds \le sup_{s\in[0,\infty)}|m_1(t,s)| \int_a^\infty [\lambda G(p,p,u) \\ &\quad + \mu G(q,q,v)]ds \\ &\quad + sup_{s\in[0,\infty)}|m_2(t,s) \int_a^\infty m_2(t,s)[(\lambda G(q,q,v) \\ &\quad + \mu G(p,p,u)]ds\end{aligned}$$

$$\begin{aligned}&2\int_a^\infty |F(p,q)(t) \\ &\quad - F(u,v)(t)|dt \le \int_a^\infty sup_{s\in[0,\infty)}|m_1(t,s)|[\lambda G(p,p,u) \\ &\quad + \mu G(q,q,v)]dt + \int_a^\infty sup_{s\in[0,\infty)}|m_2(t,s)|[(\lambda G(q,q,v) \\ &\quad + \mu G(p,p,u)]dt \le \frac{M_1}{2}[\lambda G(p,p,u) + \mu G(q,q,v)] \\ &\quad + \frac{M_2}{2}[(\lambda G(q,q,v) + \mu G(p,p,u)]\end{aligned}$$

$$\begin{aligned}&G(F(p,q), F(p,q), F(u,v)) \le \frac{M_1}{2}[\lambda G(p,p,u) \\ &\quad + \mu G(q,q,v)] + \frac{M_2}{2}[(\lambda G(q,q,v) + \mu G(p,p,u)]\end{aligned} \tag{5.2}$$

Similarly, we have

$$\begin{aligned}&G(F(q,p), F(q,p), F(v,u)) \le \frac{M_1}{2}[\lambda G(q,q,v) \\ &\quad + \mu G(q,q,u)] + \frac{M_2}{2}[(\lambda G(p,p,u) + \mu G(q,q,v)].\end{aligned} \tag{5.3}$$

Adding (5.2) and (5.3), we have

$$G(F(p,q),F(p,q),F(u,v)) + G(F(q,p),F(q,p),F(v,u)) \\ \leq \frac{(M_1+M_2)(\lambda+\mu)}{2}(G(p,p,u)+G(q,q,v))ds.$$

Again if (c, w) be a coupled lower and upper solution of the integral equation (5.1), then $c \leq F(c,w)$ and $w \geq F(w,c)$, which show that every condition of Corollary 4.2 are satisfied by taking $g = I$. By an application of Corollary 4.2 it follows that (5.1) has a solution in $L[a,\infty)$. □

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